非自励遅れ微分方程式の厳密解と振幅拡大現象

大平徹

名古屋歳学大学院多元数理科学研究科

あらまし 最近, 単純な非自励系の遅れ微分方程式において, ヒッ書の共同研究者により, 厳密解を求めることに成功した研究がある. ここでは, この遅れ微分方程式を適切なパラメータにおいて相互作用をさせると, させない場合に比べて 10^8 にも及ぶ振幅の拡大現象が出現したので報告する.

キーワード 遅れ微分方程式, 厳密解, 相互作用, 振幅拡大現象

An exact solution of a non-autonomous delay differential equation and amplitude enhancement phenomena

Toru Ohira

Graduate School of Mathematics, Nagoya University

Abstract Recently, one of the coauthors of this paper succeeded in obtaining an exact solution for a simple non-autonomous delay differential equation. Here, we report that when this delay differential equation is made to interact under appropriate parameter settings, an amplitude enhancement phenomenon occurs, with the amplitude increasing by as much as 10^8 compared to the non-interacting case.

Key words delay differential equation, exact solutio, coupling interaction, amplitude enhancement

In this study, we focus on the following non-autonomous delay differential equation:

$$\frac{dX(t)}{dt} + atX(t) = bX(t-\tau). \tag{1}$$

where a and b are real constants, and τdenotes the delay. This is considered to be the simplest form of a non-autonomous delay differential equation. In general, non-autonomous delay differential equations are recognized as being more difficult to handle than autonomous ones.

However, my collaborator and I adopted a unique approach to this particular case given by Eq. (1):

(a) Instead of treating it as a standard initial function problem, we sought solutions defined over the entire real time axis $t \in R$.

(b) While Laplace transforms are usually employed in delay differential equation analysis, we applied the Fourier transform instead [1].

(c) By appropriately anticipating the form of the solution during the inverse Fourier transform process, we developed a method to derive an exact analytical solution.

As a result, we successfully obtained the following exact solution valid for $-\infty < t < \infty$ [2].

$$X(t) = C\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{b}{a\tau}\right)^n e^{-\frac{1}{2}a(t-n\tau)^2}. \tag{2}$$

where $C$is an integration constant. Physically, this exact solution can be viewed as a superposition of Gaussian functions centered at $t = n\tau (n = 0, 1, 2, \ldots)$, each taking the maximum value $C\frac{1}{n!}\left(\frac{b}{a\tau}\right)^n$ To our knowledge, this is the first exact solution derived for a non-autonomous delay differential equation of this kind.

From a phenomenological perspective, Eq. (1) exhibits what we call a "delay-induced frequency resonance", a dynamic behavior in which certain frequencies are amplified depending on the delay parameters [3, 4].

We have further extended the equation to a two-dimensional system, where two variables can interact. The equations for the non-interacting and interacting cases are given below:

$$\frac{dX(t)}{dt} + atX(t) = bX(t - \tau_1),$$
$$\frac{dY(t)}{dt} + \alpha tY(t) = \beta Y(t - \tau_2). \tag{3}$$

$$\frac{dX(t)}{dt} + atX(t) = bY(t - \tau_1),$$
$$\frac{dY(t)}{dt} + \alpha tY(t) = \beta X(t - \tau_2). \tag{4}$$

By properly introducing and tuning the interaction with delay, we discovered a striking phenomenon: the amplitude of the dynamical trajectories can be enhanced by a factor of up to 10^8 compared to the non-interacting case (Fig. 1) [5].

This suggests that, to achieve collective oscillatory behavior with significant amplitude, it may not be necessary to have a large number of interacting elements. For example, the number of cells that generate the heart's rhythm is estimated to be about ten thousand, but our results imply that, with properly tuned delayed interactions, the same level of oscillatory amplitude could be achieved with fewer cells.

At present, this finding is based on numerical simulations, but we believe that by further deriving exact solutions for the two-dimensional case, we can deepen our understanding of the phenomenon and explore potential applications in amplitude control and enhancement.

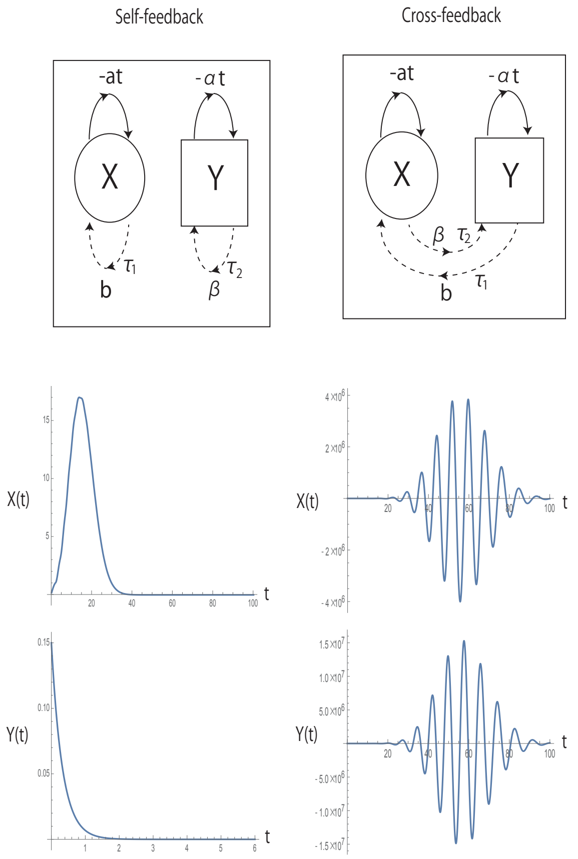

図 1: Schematic representation and example of dynamics for the (left) non-interacting case [Eq. (3)] and (right) interacting case [Eq. (4)].

参考文献

[1] K. Ohira and T. Ohira, "Solving a Delay Differential Equation through Fourier Transform," *Physics Letters A* 531, 130138 (2025).

[2] K. Ohira, "An exact solution for a non-autonomous delay differential equations," arXiv:2411.11402 (accepted and to appear in *SIAM J. on Applied Dynamical Systems*), (2024).

[3] K. Ohira, "Resonating Delay Equation," *EPL*, 137, 23001 (2022).

[4] K. Ohira, "Delayed Dynamics with Transient Resonating Oscillations," *Journal of the Physical Society of Japan*, 92, 064002, (2023).

[5] K. Ohira, T. Ohira and H. Ohira "Amplitude enhancements through rewiring of a non-autonomous delay system," *Chaos* 53, 041101 (2025).